CAPACITY OF MISMATCHED GAUSSIAN CHANNELS WITH AND WITHOUT FEEDBACK

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LISS-42

July 1989

Published in *Probab. Th. Rel. Fields* 84, 453-471 (1990)

Research partially supported by ONR Contract N00014-84-C-0212.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for Public Release: Distribution Unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Department of Statistics 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State and ZIP Code) University of North Carolina Chapel Hill, North Carolina 27514	7b. ADDRESS (City, State and ZIP Code)
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Office of Naval Research 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER ONR Contract N00014-84-C-0212
8c. ADDRESS (City, State and ZIP Code) Statistics & Probability Program Arlington, VA 22217	10. SOURCE OF FUNDING NOS. PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO. / WORK UNIT NO.

11. TITLE (Include Security Classification)
Capacity of Mismatched Gaussian Channels with and without Feedback

12. PERSONAL AUTHOR(S)
S. Ihara

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
TECHNICAL	FROM ______ TO ______	1989 July	19

16. SUPPLEMENTARY NOTATION

17. COSATI CODES		
FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
Gaussian channels, Shannon theory, stationary channels, channel capacity

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☒ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION
22a. NAME OF RESPONSIBLE INDIVIDUAL C.R. Baker	22b. TELEPHONE NUMBER (Include Area Code) (919) 962-2189 22c. OFFICE SYMBOL

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

Probab. Th. Rel. Fields 84, 453–471 (1990)

Probability Theory and Related Fields

Capacity of Mismatched Gaussian Channels with and without Feedback

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Summary. Continuous time communication channels with additive noise are considered under an average power constraint. The noises are assumed to be Gaussian processes equivalent (or mutually absolutely continuous) to a Brownian motion. We study the problem whether the capacity of the channel is increased by feedback or not. It is given a sufficient condition under which the capacity is not increased by feedback. It is also given an example of a channel whose capacity is doubled by feedback.

1. Introduction

Whether the capacity can be increased by feedback or not has been studied for various communication channels [4, 7, 13, 19, 20, 25]. Shannon [25] showed that while the coding capacity of a discrete memoryless channel with feedback is equal to that of the same channel without feedback, the zero error capacity is increased by feedback. Kadota et al. [19] showed that feedback can not increase the information capacity of the white Gaussian channel (WGC). This result has been generalized by Hitsuda and the author [13]. On the other hand it has been known that, if a Gaussian channel (GC) is with a non white noise, the capacity is increased by feedback (see [4, 24]). Moreover it was claimed by Ebert [7] and Pinsker that the capacity C^f of a GC with feedback is at most twice of the capacity C of the same channel without feedback:

$$C^f \leqq 2C. \tag{1.1}$$

In this paper we consider a continuous time GC. We give a sufficient condition under which the capacity is not increased by feedback. We also give an example of a GC whose capacity is doubled by feedback.

We are concerning the effect of feedback on the capacity of a continuous time GC presented by

$$Y(t) = \int_0^t x(u)\,du + Z(t), \qquad 0 \leqq t \leqq T, \tag{1.2}$$

* Research partially supported by ONR contract N00014-84-0212

where $x(\cdot)$, $Y(\cdot)$ and $Z(\cdot)$ are the channel input, the channel output and the noise, respectively. The noise $Z(\cdot)$ is assumed to be a Gaussian process given by

$$Z(t)=B(t)+\int_0^t \int_0^s f(s,u)\,dB(u)\,ds, \tag{1.3}$$

where $B(\cdot)$ is a Brownian motion and $f(s,u)\in L^2([0,T]^2)$ is a Volterra function (i.e., $f(s,u)=0$ if $s<u$). It is assumed that the channel is with noiseless feedback, so that the channel input $x(\cdot)$ is a causal function of the message to be transmitted and the channel output. The WGC is presented by

$$Y(t)=\int_0^t x(u)\,du+B(t), \qquad 0\leqq t\leqq T, \tag{1.4}$$

and is a special case of the GC (1.2). We assume that an average power constraint

$$\int_0^T E[x(u)^2]\,du\leqq PT \tag{1.5}$$

is imposed on the channel input, where $P>0$ is a constant. The definition of capacity used in this paper is the mutual information version, and the capacity is sometimes called the information capacity. We define the capacity as the supremum of the mutual information between the message and the channel output taken over all messages and channel inputs satisfying the constraint.

Let F and F^* be the integral operators on $L^2[0,T]$ with integral kernel $f(s,u)$ and $f^*(s,u)\equiv f(u,s)$, respectively, and define a self-adjoint operator S by

$$S=F+F^*+FF^*. \tag{1.6}$$

It is shown that if S is non-negative definite then the capacity of the GC (1.2) subject to (1.5) is not increased by feedback and is equal to $PT/2$ (Theorem 2). Although it may be expected that if S is not non-negative definite then the capacity is increased by feedback, we have not succeeded to prove.

To show that the capacity is increased by feedback, we consider a special case of the GC (1.2) where the noise $Z(\cdot)$ is given by

$$Z(t)=B(t)-\int_0^t \int_0^s e^{u-s}\,dB(u)\,ds. \tag{1.7}$$

It is shown that, if the power P is equal to 1/2, the per unit time capacity of the GC given by (1.2) and (1.7) with feedback is *at least twice* of that of the same channel without feedback (Theorem 3).

We should here give comments on relevant works appeared after this paper was submitted. For the discrete time GC, Cover and Pombra [5] proved the

inequality (1.1) with the aid of certain matrix inequalities. The author [17] showed that, for any $\varepsilon > 0$, there exists a discrete time GC for which an inequality

$$C^f > (2-\varepsilon)\, C$$

holds. Note that the noise process considered in [17] is derived from a discrete time approximation of the process $Z(\cdot)$ of (1.7). Thus we can say that the factor two in (1.1) can not be replaced by any other constants less than two. In [3] and [18], some conditions are given for the discrete time GC under which the capacity is increased by feedback.

2. Preparation

Let ξ and η be random variables defined on a probability space $(\Omega, \mathscr{B}, P)$ taking values on measurable spaces $(G, \mathscr{G})$ and $(H, \mathscr{H})$, respectively, and denote by μ_ξ and $\mu_{\xi\eta}$ the probability distribution of ξ and the joint probability distribution of ξ and η, respectively. The mutual information $I(\xi, \eta)$ between ξ and η is defined by

$$I(\xi, \eta) = \int_{G \times H} \log(d\mu_{\xi\eta}/d\mu_\xi \times \mu_\eta)\, d\mu_{\xi\eta},$$

if $\mu_{\xi\eta}$ is absolutely continuous with respect to the product measure $\mu_\xi \times \mu_\eta$ ($\mu_{\xi\eta} \ll \mu_\xi \times \mu_\eta$), where $d\mu_{\xi\eta}/d\mu_\xi \times \mu_\eta$ is the Radon-Nikodym derivative; otherwise $I(\xi, \eta)$ is infinite. Since the measurable spaces $(G, \mathscr{G})$ and $(H, \mathscr{H})$ can be taken arbitrarily, ξ and η may be stochastic processes as well as finite dimensional random variables. The conditional mutual information $I(\xi, \eta \mid \zeta)$ between ξ and η given ζ is defined by

$$I(\xi, \eta \mid \zeta) = \iint \log(d\mu_{\xi\eta|\zeta}/d\mu_{\xi|\zeta} \times \mu_{\eta|\zeta})\, d\mu_{\xi\eta|\zeta}\, d\mu_\zeta,$$

if $\mu_{\xi\eta|\zeta} \ll \mu_{\xi|\zeta} \times \mu_{\eta|\zeta}$ (μ_ζ-a.s.), where $\mu_{\xi|\zeta}$ is the conditional probability distribution of ξ given ζ, and $\mu_{\xi\eta|\zeta}$ is the conditional joint probability distribution; otherwise $I(\xi, \eta \mid \zeta)$ is infinite.

We consider the GC presented by (1.2) and (1.3). The Brownian motion $B(\cdot)$ is assumed given on $(\Omega, \mathscr{B}, P)$. Throughout the paper we assume that the feedback is instantaneous and noiseless. Precisely speaking, the following conditions are satisfied.

(a.1) The message θ to be transmitted is a random variable defined on $(\Omega, \mathscr{B}, P)$, independent of the channel noise $Z(\cdot)$, taking values on an arbitrary measurable space $(G, \mathscr{G})$.

(a.2) $x(t)$ is $\mathscr{F}(\theta) \vee \mathscr{F}_t(Y)$ measurable, where $\mathscr{F}(\theta)$ (resp. $\mathscr{F}_t(Y)$) is the σ-field generated by θ (resp. $\{Y(u); u < t\}$) and $\mathscr{F}(\theta) \vee \mathscr{F}_t(Y)$ is the smallest σ-field containing $\mathscr{F}(\theta)$ and $\mathscr{F}_t(Y)$.

(a.3) The stochastic Eq. (1.2) has a unique solution $Y(\cdot)$.

The GC is said to be without feedback, if (a.1) and following (a.2′) are satisfied:

(a.2′) $x(t)$ is $\mathscr{F}(\theta)$ measurable.

Denote by $I_T(\theta, Y) = I(\theta, Y_0^T)$ the mutual information between the message θ and the output $Y_0^T = \{Y(t); 0 \leqq t \leqq T\}$. Then, under the constraint (1.5), the (information) *capacity* $C_T^f(P)$ *of the GC* (1.2) *with feedback* is defined by

$$C_T^f(P) = \sup_{\theta, x} I_T(\theta, Y), \tag{2.1}$$

where the supremum is taken for all pairs (θ, x) of a message and an input $x(\cdot)$ satisfying (a.1)–(a.3) and (1.5). In the same manner the *capacity* $C_T(P)$ *of the GC without feedback* is defined, by taking the supremum in (2.1) for all pairs (θ, x) satisfying (a.1), (a.2′) and (1.5). Equivalently, the capacity $C_T(P)$ is given by

$$C_T(P) = \sup_x I_T(x, Y),$$

where $I_T(x, Y) = I(x_0^T, Y_0^T)$ and the supremum is taken for all inputs $x(\cdot)$ which are independent of $Z(\cdot)$ and satisfy (1.5). We are also interested in the capacity per unit time, under the constraint

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T E[x(u)^2]\, du \leqq P. \tag{2.2}$$

The *per unit time capacity* $\bar{C}^f(P)$ *of the GC* (1.2) *with feedback* is defined by

$$\bar{C}^f(P) = \sup_{\theta, x} \bar{I}(\theta, Y), \tag{2.3}$$

where

$$\bar{I}(\theta, Y) = \limsup_{T \to \infty} \frac{1}{T} I_T(\theta, Y) \tag{2.4}$$

is the per unit time mutual information and the supremum is taken for all pairs (θ, x) satisfying (a.1)–(a.3), for all $T > 0$, and (2.2). The *per unit time capacity* $\bar{C}(P)$ *of the GC without feedback* is defined in the same way.

3. Channel Whose Capacity is not Increased by Feedback

The GC presented by (1.2) and (1.3) is considered in this section. It is well known that there exists a Volterra kernel $g(s, u) \in L^2([0, T]^2)$, called the resolvent kernel of $f(s, u)$, such that

$$\begin{aligned} &f(s, u) + g(s, u) + \int_u^s f(s, v)\, g(v, u)\, dv \\ &= f(s, u) + g(s, u) + \int_u^s g(s, v) f(v, u)\, dv = 0, \qquad s, u \in [0, T], \end{aligned} \tag{3.1}$$

(see [26]). Denoting by G the integral operator on $L^2[0, T]$ with $g(s, u)$ as the kernel, (3.1) means that

$$(I+F)(I+G)=(I+G)(I+F)=I, \tag{3.2}$$

where I is the identity operator on $L^2[0, T]$. It follows from (1.3) and (3.1) that

$$Z(t)=\int_0^t F(t, u)\, dB(u), \tag{3.3}$$

$$B(t)=\int_0^t G(t, u)\, dZ(u), \tag{3.4}$$

where $F(t, u)=1+\int_u^t f(s, u)\, ds$ for $t \geqq u$, $F(t, u)=0$ for $t<u$, and $G(t, u)$ is corresponding to $g(s, u)$ in the same way (see [12]). Since (3.3) is the canonical representation of $Z(\cdot)$ in the sense of Lévy-Hida-Cramér, we can apply a result in [13] to get a formula for the mutual information in the GC.

Theorem 1. *Suppose that* $\int_0^T E[x(u)^2]\, du<\infty$. *Then the mutual information in the GC* (1.2) *is given by*

$$I_t(\theta, Y)=\tfrac{1}{2}\int_0^t E[(x_0(u)-\hat{x}_0(u))^2]\, du, \qquad t>0, \tag{3.5}$$

where $x_0=(I+G)\,x$, *more precisely,*

$$x_0(t)=x(t)+\int_0^t g(t, u)\, x(u)\, du, \qquad t>0, \tag{3.6}$$

and $\hat{x}_0(t)=E[x_0(t)\,|\,\mathscr{F}_t(Y)]$ *is the conditional expectation.*

We now can give lower and upper bounds of the capacity and also a sufficient condition under which the capacity does not change with feedback.

Theorem 2 [16]. (1) *The capacity of the GC* (1.2) *subject to* (1.5) *is bounded by*

$$PT/2 \leqq C_T(P) \leqq C_T^f(P) \leqq \|(I+F)^{-1}\|^2\, PT/2, \tag{3.7}$$

where $\|\cdot\|$ *is the operator norm of* $L^2[0, T]$.

(2) *If the operator* S *defined by* (1.6) *is non-negative definite* ($S \geqq 0$), *then*

$$C_T(P)=C_T^f(P)=PT/2. \tag{3.8}$$

Proof. (1) The second inequality of (3.7) is clear by definition. For an integer N, denote by Δ_n $(n=1, \ldots, N)$ the interval $((n-1)\,T/N, nT/N]$ and let $\theta(n)$, $n=1, \ldots, N$, be mutually independent Gaussian random variables with mean zero

and variance P. Let $\theta_N = (\theta(1), \ldots, \theta(N))$ be independent of $Z(\cdot)$ and define an input $x_N(\cdot)$ by

$$x_N(t) = \theta(n) \quad \text{if } t \in \Delta_n.$$

Then it is clear that (θ_N, x_N) satisfies (a.1), (a.2′) and (1.5). For any $\varepsilon > 0$, we can show that there exists an integer N such that

$$I_T(\theta_N, Y_N) \geqq PT/2 - \varepsilon,$$

where $Y_N(\cdot)$ is the output corresponding to (θ_N, x_N). This implies that the first inequality of (3.7) is true. This inequality was also shown by Baker [1, 2]. Let (θ, x) be any pair satisfying (a.1)–(a.3) and (1.5). Define $x_0(\cdot)$ by (3.6) and denote by $\|\cdot\|_2$ the norm of $L^2[0, T]$. Then, using Theorem 1 and (3.2), we have

$$\begin{aligned} I_T(\theta, Y) &= \tfrac{1}{2} \int_0^T E[(x_0(u) - \hat{x}_0(u))^2]\, du \\ &\leqq \tfrac{1}{2} E[\|x_0\|_2^2] = \tfrac{1}{2} E[\|(I+G)x\|_2^2] \\ &\leqq \tfrac{1}{2} \|I+G\|^2 E[\|x\|_2^2] \leqq \tfrac{1}{2} \|(I+F)^{-1}\|^2 PT. \end{aligned} \tag{3.9}$$

The last inequality of (3.7) follows from (3.9).

(2) Since S is a Hilbert-Schmidt operator, if $S \geqq 0$ then we can easily show that

$$\|(I+F)^{-1}\|^2 = \|(I+S)^{-1}\| = 1. \tag{3.10}$$

The Eq. (3.8) is straightforward of (3.7) and (3.10). □

4. Channel Whose Capacity is Increased by Feedback

Our main aim is to show that feedback increases the capacity of a GC (1.2) with the Gaussian noise $Z(\cdot)$ given by (1.7). We consider the GC on the time interval $[0, \infty)$ and under the constraint (2.2) with $P = 1/2$. We can show that the per unit time capacity of the GC with feedback is equal or greater than twice of the capacity of the same channel without feedback.

Theorem 3. *Let $\bar{C}^f(P)$ and $\bar{C}(P)$ be the per unit time capacity with and without feedback, respectively, of the GC given by* (1.2) *and* (1.7) *under the constraint* (2.2). *Then*

$$\bar{C}^f(\tfrac{1}{2}) \geqq 1 \tag{4.1}$$

and

$$\bar{C}(\tfrac{1}{2}) = \tfrac{1}{2}. \tag{4.2}$$

Let us outline how to prove Theorem 3. To show (4.1) we construct a coding scheme (θ, x), in the following way, by which mutual information $\bar{I}(\theta, Y) = 1$

is transmitted per unit time. Let θ be a Gaussian random variable independent of $Z(\cdot)$ with mean zero and variance one, and $A(t)$ be a function defined by

$$A(t)=\frac{1}{\sqrt{2}}\exp\left(\tfrac{1}{2}\int_0^t g(u)^2\,du\right),\qquad t\geqq 0, \tag{4.3}$$

where the function $g(t)$ is the unique solution of the differential equation

$$\begin{cases} 2g'(t)=-g(t)^3+\dfrac{1}{\sqrt{2}}\,g(t)^2+\sqrt{2}, & t>0,\\ g(0)=\dfrac{1}{\sqrt{2}}. \end{cases} \tag{4.4}$$

The coding scheme of the information transmission is given by

$$Y^*(t)=\int_0^t A(u)(\theta-\hat{\theta}(u))\,du+Z(t),\qquad t\geqq 0, \tag{4.5}$$

where $\hat{\theta}(u)=E[\theta\,|\,\mathscr{F}_u(Y^*)]$. Then we can prove the following proposition.

Proposition 1. (1) *The stochastic Eq.* (4.5) *has a unique solution* $Y^*(\cdot)$.

(2) $A(t)^2\,E[(\theta-\hat{\theta}(t))^2]=\frac{1}{2}$, $t\geqq 0$.

(3) $\bar{I}(\theta, Y^*)=1$.

It is clear from (1) and (2) that (θ, x) satisfies (a.1)–(a.3) and (2.2) with $P=1/2$, where $x(t)=A(t)(\theta-\hat{\theta}(t))$. Hence (4.1) follows from (3): $\bar{C}^f(1/2)\geqq\bar{I}(\theta, Y^*)=1$.

We turn to the calculation of the capacity $\bar{C}(P)$ without feedback. In place of the GC (1.2) we consider a slightly modified GC given by

$$Y_0(t)=\int_0^t x(u)\,du+Z_0(t), \tag{4.6}$$

where the noise $Z_0(\cdot)$ is a Gaussian process defined by

$$\begin{aligned} Z_0(t)&=B(t)-\int_0^t\int_{-\infty}^s e^{u-s}\,dB(u)\,ds\\ &=Z(t)-\int_0^t e^{-s}\xi_0\,ds, \end{aligned} \tag{4.7}$$

here $\xi_0=\int_{-\infty}^0 e^u\,dB(u)$ and the Brownian motion has been extended to a Gaussian process $\{B(t);\,-\infty<t<\infty\}$ in such a way that $B(\cdot)$ is with independent increments such that $E[(B(t)-B(s))^2]=|t-s|$. It is shown that the derivative $\dot{Z}_0(\cdot)$ of the process $Z_0(\cdot)$ can be regarded as a generalized stationary Gaussian process [10]. Let $\mathscr{D}$ be the space of all infinitely differentiable real functions with compact

support and $\mathscr{D}_T=\{\phi\in\mathscr{D};\operatorname{supp}(\phi)\subset[0,T]\}$. Precisely speaking, the derivative $\dot{Z}(\cdot)$ of $Z(\cdot)$ is defined by

$$\dot{Z}(\phi)=-Z(\phi')=-\int_{-\infty}^{\infty} Z(t)\,\phi'(t)\,dt,\qquad \phi\in\mathscr{D}.$$

Note that if $X(t)=\int_0^t x(u)\,du$ then $\dot{X}(\phi)=x(\phi)$. It will be shown in Lemma 4 that $\dot{Z}_0(\cdot)$ is a generalized stationary process with spectral density function (SDF)

$$f(\lambda)=\frac{\lambda^2}{2\pi(\lambda^2+1)}. \tag{4.8}$$

Differentiating the both sides of (4.6) we get a generalized stationary GC

$$\dot{Y}_0(\phi)=x(\phi)+\dot{Z}_0(\phi),\qquad \phi\in\mathscr{D}. \tag{4.9}$$

Replacing $I_T(\theta, Y)$ in (2.4) by

$$I_T(x,\dot{Y}_0)=I(\{x(\phi);\ \phi\in\mathscr{D}_T\},\{\dot{Y}_0(\psi);\ \psi\in\mathscr{D}_T\}),$$

defines the per unit time mutual information $\bar{I}(x,\dot{Y}_0)$ in the GC (4.9). Denote by $\bar{C}_0(P)$ the per unit time capacity of the GC (4.9) without feedback under the constraint (2.2). Using a result in [23] we can calculate the capacity $\bar{C}_0(1/2)$ and (4.3) follows from the following proposition.

Proposition 2. *It holds that*

$$\bar{C}(\tfrac{1}{2})=\bar{C}_0(\tfrac{1}{2})=\tfrac{1}{2}. \tag{4.10}$$

5. Proof of Propositions

We prepare some lemmas to prove Proposition 1.

Lemma 1. *The unique solution $g(t)$ of (4.4) is given by*

$$(g(t)-\sqrt{2})(g(t)-\alpha)^{\beta}(g(t)-\bar{\alpha})^{\bar{\beta}}=\gamma e^{-2t},\qquad t\geqq 0, \tag{5.1}$$

where

$$\alpha=(-1+\sqrt{7}\,i)/(2\sqrt{2}),\qquad \beta=(-\sqrt{7}+5\,i)/(2\sqrt{7}),$$

$$\gamma=-(1-\sqrt{2}\,\alpha)^{\beta}(1-\sqrt{2}\,\bar{\alpha})^{\bar{\beta}},$$

and $\bar{\alpha}$ denotes the complex conjugate of α. Moreover, it holds that

$$\lim_{t\to\infty} g(t)=\sqrt{2}. \tag{5.2}$$

Proof. The roots of the polynomial

$$Q(t)=t^3-\frac{1}{\sqrt{2}}t^2-\sqrt{2}$$

are $\sqrt{2}$, α and $\bar{\alpha}$, and it holds that

$$\frac{1}{Q(t)}=\frac{1}{4}\left(\frac{1}{t-\sqrt{2}}+\frac{\beta}{t-\alpha}+\frac{\bar{\beta}}{t-\bar{\alpha}}\right).$$

Then it is known that the solution $g(t)$ of (4.4) is given by (5.1). We put $\phi(t)=\arg(g(t)-\alpha)$. Then we have

$$\begin{aligned}(g(t)-\alpha)^{\beta}(g(t)-\bar{\alpha})^{\bar{\beta}}&=|g(t)-\alpha|^{2\operatorname{Re}(\beta)}\exp(-2\operatorname{Im}(\beta)\,\phi(t))\\&=|g(t)-\alpha|^{-1}\exp\left(-\frac{5}{\sqrt{7}}\,\phi(t)\right).\end{aligned}\tag{5.3}$$

Since the right hand side of (5.1) tends to zero as $t\to\infty$, noting (5.3) and $-\pi\leqq\phi(t)\leqq\pi$, we have

$$\lim_{t\to\infty}(g(t)-\sqrt{2})\,|g(t)-\alpha|^{-1}=0.$$

This yields (5.2). $\square$

Since the resolvent kernel $g(s,u)$ of the Volterra kernel $f(s,u)=-e^{u-s}\,(s\geqq u)$ is given by

$$g(s,u)=1,\qquad s\geqq u,$$

the expressions (3.3) and (3.4) for the process $Z(\cdot)$ of (1.7) turn to

$$Z(t)=\int_0^t e^{u-t}\,dB(u),\tag{5.4}$$

$$B(t)=Z(t)+\int_0^t Z(u)\,du.\tag{5.5}$$

We consider the following information transmission over the GC given by (1.2) and (1.7):

$$Y(t)=\int_0^t A(u)(\theta-\gamma(u))\,du+Z(t),\tag{5.6}$$

where θ is the same random variable as in (4.5) and $\eta(u)$ is $\mathscr{F}_u(Y)$ measurable. We define a function $H(t)$ and processes $\zeta(\cdot)$ and $U(\cdot)$ by

$$H(t)=A(t)+\int_0^t A(u)\,du, \tag{5.7}$$

$$H(t)\,\zeta(t)=A(t)\,\eta(t)+\int_0^t A(u)\,\eta(u)\,du, \tag{5.8}$$

$$U(t)=Y(t)+\int_0^t Y(u)\,du. \tag{5.9}$$

Using (5.4), (5.5) and (5.9) we can easily show the following relations.

$$U(t)=\int_0^t H(u)(\theta-\zeta(u))\,du+B(t), \tag{5.10}$$

$$Y(t)=\int_0^t e^{u-t}\,dU(u). \tag{5.11}$$

It is clear from (5.9) and (5.11) that

$$\mathscr{F}_t(Y)=\mathscr{F}_t(U), \qquad t\geqq 0, \tag{5.12}$$

meaning $I_t(\theta, Y)=I_t(\theta, U)$ for all $t\geqq 0$. Since $\zeta(u)$ is $\mathscr{F}_u(Y)=\mathscr{F}_u(U)$ measurable, (5.10) represents a WGC with feedback. The formula for the mutual information in the WGC has been known (see, e.g., [22], Chap. 16).

Lemma 2. *The mean square filtering error in the GC's* (5.6) *and* (5.10) *is given by*

$$E[(\theta-\hat{\theta}(t))^2]=\left(1+\int_0^t H(u)^2\,du\right)^{-1}, \tag{5.13}$$

where $\hat{\theta}(t)=E[\theta\,|\,\mathscr{F}_t(Y)]=E[\theta\,|\,\mathscr{F}_t(U)]$. *The mutual information is given by*

$$\begin{aligned} I_T(\theta, Y)=I_T(\theta, U)&=\tfrac{1}{2}\int_0^T H(t)^2\,E[(\theta-\hat{\theta}(t))^2]\,dt \\ &=\tfrac{1}{2}\int_0^T H(t)^2\left(1+\int_0^t H(u)^2\,du\right)^{-1} dt. \end{aligned} \tag{5.14}$$

Note that the resulting mutual information does not depend on $\eta(\cdot)$. Now we can prove Proposition 1.

Proof of Proposition 1. (1) Define processes $U_0(\cdot)$ and $\eta(\cdot)$ by

$$U_0(t) = \int_0^t H(u)\theta\, du + B(t),$$

$$\eta(t) = E[\theta \mid \mathscr{F}_t(U_0)]. \tag{5.15}$$

Then $H(t)\zeta(t)$ of (5.8) can be written in the form $H(t)\zeta(t) = \int_0^t h(t, s)\, dU_0(s)$ with an L^2-Volterra kernel $h(t, s)$. It can be shown ([15]) that the stochastic equation

$$U(t) = U_0(t) - \int_0^t \int_0^u k(u, s)\, dU(s)\, du, \tag{5.16}$$

where $k(t, s)$ is the resolvent of $h(t, s)$, has a unique solution

$$U(t) = U_0(t) + \int_0^t \int_0^u h(u, s)\, dU_0(s)\, du. \tag{5.17}$$

In other words, (5.10) has a unique solution $U(\cdot)$ when $H(u)\zeta(u) = \int_0^u h(u, s)\, dU_0(s) = \int_0^u k(u, s)\, dU(s)$. Since there is one-to-one correspondence between (5.6) and (5.10), this means that (5.6) has a unique solution $Y(\cdot)$ when $\eta(\cdot)$ is given by (5.15). From (5.16), (5.17) and (5.12) we see that $\mathscr{F}_t(U_0) = \mathscr{F}_t(U) = \mathscr{F}_t(Y)$ and that $\eta(t) = E[\theta \mid \mathscr{F}_t(U_0)] = E[\theta \mid \mathscr{F}_t(Y)]$. Thus (4.5) has a unique solution $Y^*(\cdot)$.

(2) It follows from (4.4) that

$$\begin{aligned}
&\frac{d}{dt}\left\{g(t)\exp\left(\tfrac{1}{2}\int_0^t g(u)^2\, du\right)\right\} \\
&= (g'(t) + \tfrac{1}{2} g(t)^3)\exp\left(\tfrac{1}{2}\int_0^t g(u)^2\, du\right) \\
&= \frac{1}{\sqrt{2}}(\tfrac{1}{2} g(t)^2 + 1)\exp\left(\tfrac{1}{2}\int_0^t g(u)^2\, du\right) \\
&= \frac{1}{\sqrt{2}}\frac{d}{dt}\left\{\exp\left(\tfrac{1}{2}\int_0^t g(u)^2\, du\right) + \int_0^t \exp\left(\tfrac{1}{2}\int_0^u g(s)^2\, ds\right) du\right\} \\
&= \frac{d}{dt}\left(A(t) + \int_0^t A(u)\, du\right).
\end{aligned}$$

Therefore, noting (5.7) and the initial condition $g(0)=A(0)=1/\sqrt{2}$, we obtain

$$H(t)=A(t)+\int_0^t A(u)\,du=g(t)\exp\left(\tfrac{1}{2}\int_0^t g(u)^2\,du\right),$$

$$H(t)^2=g(t)^2\exp\left(\int_0^t g(u)^2\,du\right)=\frac{d}{dt}\exp\left(\int_0^t g(u)^2\,du\right) \tag{5.18}$$

and

$$1+\int_0^t H(u)^2\,du=\exp\left(\int_0^t g(u)^2\,du\right). \tag{5.19}$$

The desired equation is straightforward of (4.3), (5.13) and (5.19).

(3) It follows from (5.18), (5.19) and (5.2) that

$$\lim_{T\to\infty} H(T)^2\left(1+\int_0^T H(u)^2\,du\right)^{-1}=\lim_{T\to\infty} g(T)^2=2.$$

Hence, by Lemma 2, we get

$$\begin{aligned}\bar{I}(\theta, Y^*)&=\lim_{T\to\infty}\frac{1}{T}I_T(\theta, Y^*)\\&=\lim_{T\to\infty}\frac{1}{2T}\int_0^T H(u)^2\left(1+\int_0^u H(s)^2\,ds\right)^{-1}du\\&=\lim_{T\to\infty}\tfrac{1}{2}H(T)^2\left(1+\int_0^T H(s)^2\,ds\right)^{-1}=1. \quad \square\end{aligned}$$

We prepare two lemmas to prove Proposition 2.

Lemma 3. *Let* $x(\cdot)$ *be an input independent of* $B(\cdot)$, *and* $Y(\cdot)$, $Y_0(\cdot)$ *and* $\dot{Y}_0(\cdot)$ *be the corresponding outputs of the GC's* (1.2) (*with the noise* $Z(\cdot)$ *given by* (1.7)), (4.6) *and* (4.9), *respectively. Then*

$$I_T(x, Y_0)=I_T(x, \dot{Y}_0), \qquad T\geqq 0, \tag{5.20}$$

and

$$\bar{I}(x, Y)=\bar{I}(x, Y_0)=\bar{I}(x, \dot{Y}_0). \tag{5.21}$$

Proof. Using the same arguments as in [11], (5.20) can be shown. The second equality of (5.21) is clear from (5.20). To show the first equality of (5.21) we put

$$\xi(s)=\int_0^s e^u\,dB(u), \qquad s\geqq 0,$$

which are independent of $\xi_0 = \int_{-\infty}^{0} e^u \, dB(u)$. Note that the conditional mutual information satisfies

$$I(\xi, (\eta, \zeta)) = I(\xi, \zeta) + I(\xi, \eta \mid \zeta) = I(\xi, \eta) + I(\xi, \zeta \mid \eta).$$

Denoting $I_T(x, Y \mid \xi) = I(x_0^T, Y_0^T \mid \xi)$ and $I_T(x, \xi \mid Y) = I(x_0^T, \xi \mid Y_0^T)$, we have

$$I_T(x, (Y_0, \xi_0)) = I_T(x, \xi_0) + I_T(x, Y_0 \mid \xi_0) = I_T(x, Y)$$
$$I_T(x, (Y_0, \xi_0)) = I_T(x, Y_0) + I_T(x, \xi_0 \mid Y_0),$$

and consequently

$$I_T(x, Y) - I_T(x, Y_0) = I_T(x, \xi_0 \mid Y_0). \tag{5.22}$$

In the same way we have

$$\begin{aligned} I_T(x, \xi_0 \mid Y_0) &= I_T((x, Y_0), \xi_0) - I_T(Y_0, \xi_0) \\ &= I_T(x, \xi_0) + I_T(Y_0, \xi_0 \mid x) - I_T(Y_0, \xi_0) \\ &= I_T(Z_0, \xi_0) - I_T(Y_0, \xi_0) \leqq I_T(Z_0, \xi_0). \end{aligned} \tag{5.23}$$

Noting (4.7) we can calculate $I_T(Z_0, \xi_0)$ by use of (5.14):

$$I_T(Z_0, \xi_0) = \tfrac{1}{2} \int_0^T \frac{1}{2+t} \, dt = \tfrac{1}{2} \log\left(1 + \frac{T}{2}\right).$$

Therefore,

$$\lim_{T \to \infty} \frac{1}{T} I_T(Z_0, \xi_0) = 0. \tag{5.24}$$

The first equality of (5.21) follows from (5.22), (5.23) and (5.24).

Lemma 4. *The process $\dot{Z}_0(\cdot)$ is a generalized stationary Gaussian process with covariance functional*

$$E[\dot{Z}_0(\phi) \dot{Z}_0(\psi)] = \int_{-\infty}^{\infty} \phi(t) \psi(t) \, dt - \tfrac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-|t-s|} \phi(s) \psi(t) \, ds \, dt, \qquad \phi, \psi \in \mathcal{D}, \tag{5.25}$$

and with SDF $f(\lambda)$ of (4.8), *namely*

$$E[\dot{Z}_0(\phi) \dot{Z}_0(\psi)] = \int_{-\infty}^{\infty} \hat{\phi}(\lambda) \overline{\hat{\psi}(\lambda)} f(\lambda) \, d\lambda, \tag{5.26}$$

where $\hat{\phi}(\lambda) = \int_{-\infty}^{\infty} e^{it\lambda} \phi(t) \, dt$ is the Fourier transform of ϕ.

Proof. We put

$$\zeta(t)=\int_0^t \int_{-\infty}^s e^{u-s}\, dB(u)\, ds,$$

for simplicity. Then $Z_0(t)=B(t)-\zeta(t)$ and

$$\dot{Z}_0(\phi)=\dot{B}(\phi)-\dot{\zeta}(\phi), \qquad \phi\in\mathscr{D}. \tag{5.27}$$

It is well known that

$$E[\dot{B}(\phi)\,\dot{B}(\psi)]=\int_{-\infty}^{\infty} \phi(t)\,\psi(t)\,dt, \qquad \phi,\psi\in\mathscr{D} \tag{5.28}$$

(see [16]). Exchanging the order of the ordinary integrals, the Wiener integrals and the expectations, we can show

$$E[\dot{B}(\phi)\,\dot{\zeta}(\phi)]=\int_{-\infty}^{\infty}\int_{-\infty}^{t} e^{-(t-s)}\,\phi(s)\,\psi(t)\,ds\,dt, \tag{5.29}$$

$$E[\dot{\zeta}(\phi)\,\dot{\zeta}(\psi)]=\tfrac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-|t-s|}\,\phi(s)\,\psi(t)\,ds\,dt, \tag{5.30}$$

by elementary calculation. We can derive (5.25) from (5.27)–(5.30). Denote $\phi_\tau(t)=\phi(t-\tau)$. Then it is clear from (5.25) that

$$E[\dot{Z}_0(\phi_\tau)\,\dot{Z}_0(\psi_\tau)]=E[\dot{Z}_0(\phi)\,\dot{Z}_0(\psi)],$$

meaning that the process $\dot{Z}_0(\cdot)$ is stationary. Since

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}(\lambda)\,\overline{\hat{\psi}(\lambda)}\,(\lambda^2+1)^{-1}\,d\lambda\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(s)\,\psi(t)\,e^{i(s-t)\lambda}(\lambda^2+1)^{-1}\,d\lambda\,ds\,dt\\
&=\tfrac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-|t-s|}\,\phi(s)\,\psi(t)\,ds\,dt,
\end{aligned}$$

(5.26) follows from (4.8) and (5.25). □

We are now in a position to prove Proposition 2.

Proof of Proposition 2. It is clear from (5.21) that

$$\bar{C}(P)=\bar{C}_0(P). \tag{5.31}$$

Let R_T be the covariance operator of the process $\dot{Z}_0(\cdot)$ on the interval $[0, T]$, namely an operator on $L^2[0, T]$ such that

$$(R_T\phi, \psi)_T = \int_0^T \phi(t)\psi(t)\,dt - \tfrac{1}{2}\int_0^T\int_0^T e^{-|t-s|}\phi(s)\psi(t)\,ds\,dt,$$
$$\phi, \psi \in L^2[0, T],$$

where $(\cdot,\cdot)_T$ is the inner product of $L^2[0, T]$. Denote by K_T the integral operator on $L^2[0, T]$ having $1/2e^{-|t-s|}$ as the integral kernel. Then $R_T = I - K_T$, where I is the identity operator. We can show that the eigenvalues $\{\kappa_n(T)\}$ of K_T are

$$\kappa_n(T) = (1 + \sigma_n(T)^2)^{-1}, \quad n = 1, 2, \ldots,$$

where $\sigma_n \equiv \sigma_n(T)$ is uniquely determined by

$$\tan((n\pi - \sigma_n T)/2) = \sigma_n, \quad (n-1)\pi < \sigma_n T \leqq n\pi.$$

The corresponding eigenfunctions $\{\phi_n(t) \equiv \phi_n(t; T)\}$ are given by

$$\phi_n(t) = \exp(i\sigma_n t) - \frac{1 - i\sigma_n}{1 + i\sigma_n}\exp(-i\sigma_n t), \quad n = 1, 2, \ldots.$$

Therefore the eigenvalues $\{\lambda_n(T)\}$ of R_T are

$$\lambda_n(T) = 1 - \kappa_n(T) = \sigma_n(T)^2(1 + \sigma_n(T)^2)^{-1}, \quad n = 1, 2, \ldots.$$

Let $C_T(P)$ be the capacity of the GC (4.9) without feedback, on the finite interval $[0, T]$, subject to (1.5). Then it is known that

$$C_T(P) = \tfrac{1}{2}\sum_{n=1}^{N} \log \frac{A(T, P)}{\lambda_n(T)}, \tag{5.32}$$

where a positive constant $A = A(T, P)$ and an integer $N = N(T, P)$ are uniquely determined by

$$\sum_{n=1}^{N} (A - \lambda_n(T)) = PT \quad \text{and} \quad N = \max\{n;\, \lambda_n(T) < A\}.$$

It can be shown that

$$\frac{1}{2} - \frac{1}{T} < \frac{1}{T}\sum_{n=1}^{\infty} \kappa_n(T) < \frac{1}{2} + \frac{1}{T}.$$

Hence we know that

$$\lim_{T\to\infty} N(T, \tfrac{1}{2}) = \infty, \quad \lim_{T\to\infty} A(T, \tfrac{1}{2}) = 1. \tag{5.33}$$

It follows from (5.32) and (5.33) that

$$\lim_{T\to\infty}\frac{1}{T}C_T(\tfrac{1}{2})=\lim_{T\to\infty}\frac{1}{2T}\sum_{n=1}^{\infty}\log\frac{1}{\lambda_n(T)}$$
$$=\lim_{T\to\infty}\frac{1}{2T}\int_1^{\infty}\log(1+\pi^{-2}T^2x^{-2})\,dx$$
$$=\lim_{T\to\infty}\frac{1}{2\pi}\int_{\pi/T}^{\infty}\log(1+x^{-2})\,dx=\tfrac{1}{2}.$$

We can also show that

$$\lim_{T\to\infty}\frac{1}{T}C_T(\tfrac{1}{2}+\varepsilon)=\frac{1}{2}+\frac{\varepsilon}{2},$$

for every $\varepsilon>0$. Therefore, for any $\varepsilon>0$ and any input signal $x(\cdot)$ satisfying (2.2) with $P=1/2$, there exists T_0 such that

$$\frac{1}{T}I_T(x,\dot{Y})\leqq\frac{1}{T}C_T(\tfrac{1}{2}+\varepsilon)\leqq\tfrac{1}{2}+\varepsilon,\qquad T\geqq T_0.$$

Since ε is arbitrary, we conclude that

$$\bar{C}_0(\tfrac{1}{2})\leqq\tfrac{1}{2}. \tag{5.34}$$

Let an input signal $x(\cdot)$ of the GC (4.9) be a mean zero stationary Gaussian process with SDF

$$h(\lambda)=\frac{1}{2\pi(\lambda^2+1)}.$$

Since

$$E[x(t)^2]=\int_{-\infty}^{\infty}h(\lambda)\,d\lambda=\tfrac{1}{2},\qquad -\infty<t<\infty,$$

the input $X(\cdot)$ satisfies (2.2) with $P=1/2$. Since the SDF $h(\lambda)$ is rational, we can apply the Pinsker's formula ([23], Theorem 10.3.1) to get

$$\bar{I}(x,\dot{Y})=\frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1+\frac{h(\lambda)}{f(\lambda)}\right)d\lambda$$
$$=\frac{1}{4\pi}\int_{-\infty}^{\infty}\log(1+\lambda^{-2})\,d\lambda=\tfrac{1}{2}. \tag{5.35}$$

Therefore, we have

$$\bar{C}_0(\tfrac{1}{2})\geqq\bar{I}(x,\dot{Y})=\tfrac{1}{2}. \tag{5.36}$$

Combining (5.31), (5.34) and (5.36) we have (4.10). □

6. Remarks

There is a well known formula for the capacity $\bar{C}(P)$ of a stationary GC:

$$\bar{C}(P)=\frac{1}{4\pi}\int_{\Lambda}\log\frac{A}{f(\lambda)}\,d\lambda, \tag{6.1}$$

where $f(\lambda)$ is the SDF of the noise process, $\Lambda=\{\lambda; f(\lambda)<A\}$ and the constant A is determined by

$$\int_{\Lambda}(A-f(\lambda))\,d\lambda=P$$

(see, e.g., [8], Theorem 5.26; [9], Theorem 8.5.1). We have not applied the formula (6.1) to calculate the capacity of the GC (4.9), since the noise is not an ordinary process but a generalized one. However, the resulting capacity $\bar{C}_0(1/2)=1/2$ is equal to the right hand side of (6.1) with $f(\lambda)$ of (4.8) and $P=1/2$.

In the same manner as in the proof of Theorem 3, we can derive a lower bound for the capacity $\bar{C}^f(P)$ of the GC of Theorem 3. Let $\gamma(P)$ be the unique solution of the equation

$$x^3-P(x+1)^2=0.$$

Then

$$\bar{C}^f(P)\geqq\gamma(P). \tag{6.2}$$

It is conjectured that, for the GC of Theorem 3, the inequality

$$\bar{C}^f(P)\geqq 2\,\bar{C}(P)$$

holds only if $P=1/2$.

We note that the process $Z(\cdot)$ of (1.3) is equivalent (or mutually absolutely continuous) to a Brownian motion on each time interval $[0, T]$ ([12]). Denote by $\|\cdot\|_T$ the norm of the reproducing kernel Hilbert space corresponding to $B_0^T=\{B(t); 0\leqq t\leqq T\}$. Then the constraint (1.5) can be written in the form

$$E[\|x_0^T\|_T^2]\leqq PT, \tag{6.3}$$

where $X(t)=\int_0^t x(u)\,du$. Since the constraint (1.5) or (6.3) is given in terms of $B(\cdot)$ and not of the channel noise $Z(\cdot)$, the GC (1.2) subject to (1.5) is called a mismatched GC (see [1, 2]). On the other hand the WGC (1.4) is to be a matched GC under (1.5). In this paper we have treated GC's with noises equivalent to a Brownian motion. This is rather for technical reasons. We can investigate a matched or a mismatched GC with an arbitrary Gaussian noise [1, 2, 13]. Baker [1, 2] has determined the capacity of the mismatched GC without feedback. It has been shown that the capacity of the matched GC is not changed by feedback under a moderate assumption on the Gaussian noise [13]. Theorem 2 may be generalized by using a similar method as in [13].

Ebert [7] claimed that the inequality (1.1) holds for the GC (1.2) subject to (2.2), under an assumption that the Volterra kernel $f(s,u)$ is a function of $s-u$. If the inequality (1.1) is true for the continuous time GC, the feedback capacity $\bar{C}^f(1/2)$ of the GC of Theorem 3 would be equal to one, twice of the capacity $\bar{C}(1/2)=1/2$ without feedback, and the coding scheme given by (4.5) would be optimal in the sense of attaining the capacity. We recall that, for the WGC, a coding scheme given in the same manner as (4.5) is optimal [14].

In this paper we have dwelled only on information capacity. In case of discrete time GC, Cover and Pombra [5] proved the inequality (1.1) for the information capacity, and showed that the information capacity is achievable by a feedback code. This means that, in case of discrete time GC, the information capacity $\bar{C}^f(P)=\bar{C}^f_{\mathrm{inf}}(P)$ per unit time is equal to the coding capacity $\bar{C}_{\mathrm{cod}}(P)$. For the continuous time GC with feedback, as far as I know, such an equality has not been established. However we can show that the lower bound $\gamma(P)$ for the capacity $\bar{C}^f(P)$ (see (6.2)) of the GC of Theorem 3 is achievable by a feedback code, meaning that $\gamma(P)\leqq\bar{C}^f_{\mathrm{cod}}(P)\leqq\bar{C}^f_{\mathrm{inf}}(P)$. For the capacity without feedback we refer [6, 9, 21].

Acknowledgement. The author wishes to express his sincere thanks to Professor C.R. Baker for valuable discussions.

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Received June 4, 1986; in revised form August 4, 1989